Sensitivity of frequency-dependent traveltimes to laterally heterogeneous, anisotropic Earth structure

Li Zhao and Thomas H. Jordan
Department of Earth, Atmospheric and Planetary Sciences, Massachusetts Institute of Technology, Cambridge, MA 02139, USA

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SUMMARY
We investigate the effect of lateral heterogeneity on the frequency-dependent traveltime residuals of various seismic arrivals, for example P, S, sS, SS, ISS, SS. Love and Rayleigh waves. These residuals, which are examples of generalized seismological data functionals (GSDFs), are measured from narrow-band cross-correlagrams between observed seismograms and isolated waveforms (isolation filters) synthesized by weighted (partial) normal-mode summations. The effect of lateral heterogeneity is incorporated through the coupling between normal-mode multiplets with the help of first-order perturbation theory. Based upon the normal-mode coupling matrix for the eigen-frequency shifts, the sensitivity kernels of the frequency-dependent traveltime residuals to the model parameter perturbations are derived by an application of the Born approximation. In order to reduce the computational labour so that tomographic inversions can practically be conducted, 2-D sensitivity kernels of the traveltime residuals to the lateral structure within the source–receiver great-circle plane are obtained with a stationary-phase integration. In addition, a normal-mode coupling scheme is adopted to increase further the computational efficiency in which a pair of modes are coupled only when the differences between their eigenfrequencies and group velocities are small.

We present numerical examples for the 2-D Fréchet kernels of frequency-dependent traveltime residuals for various model parameters in a transversely isotropic model, namely the velocities of vertically and horizontally polarized and/or propagating shear and compressional waves and the topographies of the 410 and 660 km discontinuities. Wherever possible, physical explanations are also provided on the different aspects of the 2-D Fréchet kernels with emphasis on the complex phenomenon of interference among multiple seismic waves. We also demonstrate, both algebraically and numerically, that when the waveforms used in the measurements involve multiple travelling waves, the frequency-dependent traveltime kernels can be counterintuitive, i.e. velocity increase in some places can lead to increased traveltimes.

Key words: anisotropy, Fréchet derivatives, lateral heterogeneity, normal modes, perturbation methods, wave propagation.

1 INTRODUCTION
In the last decade or so, our knowledge of the Earth’s mantle structure has been greatly advanced with the accumulation of results from seismic tomography studies made possible by the availability of increasingly powerful computational facilities and various deployments of high-quality digital seismic instruments. This progress has also benefited from the theoretical development in the research of seismic wave propagation in more realistic earth models. Most of the global as well as regional tomographic studies rely on two types of quantities extracted from seismograms: the arrival times and waveforms. In the former case, seismic arrival times are collected and their residuals relative to a reference model are inverted (e.g. Inoue et al. 1990; Woodward & Masters 1991; Grand 1994; van der Hilst, Widiantoro & Engdahl 1997). Since the number of data is often very large in traveltime tomography, emphasis is on the inversion procedures, and the forward modelling of the individual records is simplified by the assumption that each of the traveltimes is accumulated during the propagation of the wave along a geometrical ray path. Under this assumption, the structural effect on the traveltime residual is uniform along
the entire ray path and vanishes everywhere away from it. In waveform tomography, on the other hand, waveforms or waveform-generated functionals obtained from a relatively smaller number (in comparison with traveltime tomography) of long-period surface- and body-wave seismograms are inverted (e.g. Woodhouse & Dziewonski 1984; Gomber et al. 1988; Zhang & Tanimoto 1989). The non-linear dependence of the waveform on the structure as well as the large number of samples from each record requires more effort on the treatment of individual seismograms. In order for the waveform tomography to be computationally practical, forward modeling is usually conducted with the understanding that the long-period waveforms are influenced by the averaged structure between the source and receiver within the great-circle plane.

Recently, as more and more relatively short-period surface and body waves are being used in waveform tomography, attempts have been made to account in a more realistic fashion for the influence of the lateral heterogeneities on the waveforms. By taking the coupling between the normal-mode multiplets into account, Li & Tanimoto (1993) developed a formalism to calculate 2-D sensitivity kernels of waveforms, which has been used in a global inversion of SH and Love waves with periods as short as 32 and 80 s, respectively (Li & Romanowicz 1996). In another approach, starting with the 2-D guided wave theory of Kennett (1984), Marquering & Snieder (1995) proposed an efficient technique for obtaining realistic 2-D sensitivity kernels of waveforms to shear-wave velocity with a surface-wave coupling formalism. Recently, 3-D kernels of waveforms have also been obtained in a similar way (Meier et al. 1997). These kernels have been combined with the method of partitioned waveform inversion (Nolet 1990) in an inversion scheme (Marquering, Snieder & Nolet 1996; Meier et al. 1997). As a novel extension of the work of Li & Tanimoto (1993), Tanimoto (1995) studied the sensitivity of the time shift of the waveform to laterally heterogeneous perturbations through a simple relationship between the time shift of a seismogram and its time derivative, or the slope of the waveform.

The purpose of the current study is to develop an alternative approach to the tomographic inversion of laterally heterogeneous structures based on the analysis of generalized seismological data functionals (GSDFs) presented by Gee & Jordan (1992). With an application of the first-order quasi-degenerate normal-mode perturbation theory (Woodhouse 1980), we derive the expressions for the Fréchet kernels of a specific type of GSDF, namely the frequency-dependent phase-delay or traveltime residual measured over a certain narrow frequency band. This type of traveltime is different from the commonly used version associated with infinite-frequency ray theory, and the waveform time shift of Tanimoto (1995). The intrinsic relation of the narrow-band cross-correlogram between the synthetic and observed seismograms to a Gaussian wavelet, discussed in detail in Gee & Jordan (1992), provides a better linear dependence of the measurement on structural perturbation. Furthermore, the frequency-dependent measurements and Fréchet kernels enable us to extract much more independent information from the same seismogram than the usual broad-band approaches, thus improving the resolving power in the inversions. In our derivation, the Born approximation is applied to obtain the expressions for the sensitivity kernels of the traveltime residuals to the model perturbations. In order to reduce the amount of computation required for inversions, we have followed the approach of Li & Tanimoto (1993) and obtained the equivalent 2-D Fréchet kernels within the source–receiver great-circle plane by carrying out a stationary-phase integration. Because they allow us to handle multiple arrivals, for example P, S, SS, sS, SS, SS, Love and Rayleigh waves, on three-component seismograms, in a consistent manner, these 2-D Fréchet kernels are very useful tools for investigating the upper-mantle lateral structure. As we shall show, they provide insights into the complicated problem of seismic wave propagation in slightly heterogeneous structures and can be used to improve the resolving power in tomographic inversions. With these Fréchet kernels, a 2-D, high-resolution image of the upper-mantle shear velocity along a central Pacific corridor between Tonga and Hawaii has been obtained (Katzman, Zhao & Jordan 1998). Among the findings from this image are the lateral harmonic patterns of high and low velocities along this corridor, whose 1500 km or so horizontal length scale strongly supports the existence of small-scale convection in the upper mantle.

In the following sections, we first provide a summary of the GSDF methodology upon which our current work is based. In Section 3, we derive the expressions for 3-D as well as the equivalent 2-D Fréchet kernels of the frequency-dependent traveltime residuals. Numerical examples for the 2-D sensitivity kernels of various types of seismic waves to different types of model parameters are presented in Section 4, along with discussions about the influence on the nature of the 2-D kernels by different factors. Details on the implementation of these Fréchet kernels in tomographic inversions will be discussed elsewhere (Katzman et al. 1998).

2 GSDF IN A SPHERICAL EARTH MODEL: A REVIEW

In the GSDF analysis, frequency-dependent phase delays are measured for selected wave groups and inverted by linearizing the functional relationship between the traveltime residuals and model perturbations. Surface waves and turning body waves are often the choices for investigating the upper-mantle structure, whereas the deeper penetrating waves, such as the core–mantle boundary (CMB) reflected ScS reverberations, can be used to study the lower mantle. In this process, almost all types of waves with drastically different propagation histories can be modelled efficiently in the same manner; the measurements are further cleared of the possible contaminations from origins other than the structure, such as the interference with non-selected waves as well as windowing and filtering effects incurred in the measurement process, to ensure the consistency between the data and the inverting kernels; and anelastic and anisotropic earth models can also be considered. Detailed accounts of the GSDF method and its implementation can be found in Gee & Jordan (1992) and Gaherty, Jordan & Gee (1996). They are briefly recalled in this section to facilitate our discussion on the extension to the laterally heterogeneous case.

Fig. 1 illustrates the GSDF analyzing process for a particular source–receiver path at an epicentral distance of 54°. The data (bottom trace) is the vertical-component seismogram recorded at station HON in Hawaii from a shallow event in the Kermadec Islands region. The full synthetic, \( \delta(t) \), is obtained from a complete normal-mode summation for the reference model PA5, a transversely isotropic and path-averaged model obtained for the Tonga–Hawaii corridor (Gaherty et al. 1996).
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normal-mode summation. The top trace is computed with only fundamental modes. Traces (a), (b) and (c) are isolation filters for Rayleigh, SS and direct S waves, respectively, obtained by weighted normal-mode summations.

Clearly visible in both the data and the synthetic are the S , SS and Rayleigh wave groups, while all the other arrivals are much weaker. For a selected wave group, a partial synthetic for the same reference model is created by the weighted normal-mode summation:

\[ f^n(t) = \sum_{j=1}^{N} w_j R_j S_j \exp(-iv_j t), \]

where \( R_j \) and \( S_j \) are the receiving and source excitation functions, described in Appendix A, of the \( j \)th mode whose eigenfrequency is \( v_j = \omega_j - i\eta_j \), with \( \omega_j \) and \( \eta_j \) the real eigenfrequency and attenuation, respectively. In the summation in (1), as in other expressions in this paper, only the real part is understood. For a full synthetic, \( w_j \equiv 1 \), and \( N \) is the total number of modes within a specified frequency band. The partial synthetic, \( f^n(t) \), is called the isolation filter; it is designed to model only the target wave group and suppress the other arrivals as much as possible. In Fig. 1, traces (a), (b) and (c) are isolation filters for the Rayleigh, SS and S waves, respectively.

The quality of the isolation filter can be assessed by the similarity of its waveform to that of the target wave in the full synthetic and its vanishing amplitude away from the target wave. A good isolation filter is achieved by carefully determining the weighting factors in the partial normal-mode summation (1). A weighting factor \( w_j \) represents the relative contribution of the \( j \)th mode to the isolation filter which contains waves with similar properties, such as group and phase velocities as well as compressional and shear energy contents. It is the interference of these waves that gives rise to the target arrival on the seismogram. For example, the Rayleigh wave group on trace (a) in Fig. 1 arrives later than the S and SS waves and is composed of mostly shear waves trapped within the shallow part of the Earth with small group and phase velocities. The S wave, on the other hand, contains shear waves penetrating deeper into the mantle with higher group and phase velocities. Owing to the correspondence between propagating seismic waves and normal modes (e.g. Ben-Menahem 1964; Gilbert 1975; Zhao & Dahlen 1993), the group and phase velocities as well as the proportions of compressional and shear energies of the target wave group can be used as the guidance in determining the weighting factors for the normal modes in the summation for an isolation filter. For example, for the SV-wave isolation filter in Fig. 1, we can find out the approximate values of its group and phase velocities based on the source depth, epicentral distance and the approximate arrival time. Group- and phase-velocity windows are then set up and the weighting factor of each mode is determined according to its modal group and phase velocities. The optimum widths of the windows are decided by a trial and error process. In the meantime, modes with a higher proportion of compressional energy are excluded for this SV-wave isolation filter. The resulting weighting factors are shown in the spheroidal-mode dispersion diagrams in Fig. 2 for the isolation filters in Fig. 1. The contributing modes are the open circles whose sizes are proportional to their weights in the summation.

For each of the normal modes in the dispersion diagram, location is an indicator of phase velocity \( c \) (or ray parameter \( p \) in the language of ray theory, since \( pc = 1 \)). The Jeans relation (Jeans 1923), \( op = 1 + 1/2 \), dictates that the farther away is a mode from the vertical axis, the smaller is its phase velocity (or the larger is its ray parameter). On the other hand, the slopes of the individual branches in the dispersion diagram provide the group velocities. In Fig. 2, one can easily observe that the branches are smoother in the region of smaller phase velocity and are more complicated elsewhere. This is explained by the fact that the modes of smaller phase velocities are closely related only to the shear waves in the mantle, and the dominance of a single type of wave leads to the simple dispersion behaviour. The modes of large phase velocities, however, correspond to both compressional and shear waves propagating deeper into the Earth. The presence of different types of waves and their mutual conversions at the internal discontinuities produce rather complicated interference patterns. In Fig. 2(c), as expected, the modes contributing to the SV-wave isolation filter fall into a phase-velocity band (a fan-shaped area). Most of the branches with the same overtone numbers are broken into a few segments by non-contributing modes whose group velocities are outside the required group-velocity window.

Fig 3 shows the information the GSDF method uses in the measurement from a transverse-component seismogram.

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Figure 2. Spheroidal-mode dispersion diagrams for earth model PA5 showing the weighting factors in the partial normal-mode summations for the respective isolation filters (a), (b) and (c) in Fig. 1. In all diagrams, the sizes of the open circles are proportional to the values of the weighting factors for the corresponding modes. The straight diagonal lines correspond to the ray parameter values at which, counter-clockwise from the right, the $S$ wave grazes the Moho, the 410-km and 660-km discontinuities, the $P$ wave grazes the 410-km discontinuity, and the $S$ wave grazes the CMB. The branch numbers $n = 0, 10, 20, 30, 40$ and 50 are indicated at the top of each diagram.
The clearly visible arrivals in both the recorded and synthetic seismograms are the direct SH and Love waves. Fig. 4 illustrates the toroidal-mode weighting factors for the Love- and SH-wave isolation filters. The contribution to the Love wave comes only from the lowest six or so branches. The branches in these toroidal-mode dispersion diagram are much smoother and therefore, unlike in the spheroidal-mode diagram in Fig. 2(e) for the SV-wave isolation filter, there are no breaks in the contributing modes along the branches.

As an example of the target wave not dominated by shear wave, the isolation filter and weighting factors of a P wave are shown in Figs 5 and 6, respectively. Because of the higher P-wave phase velocity, the contributing modes are in a band closer to the vertical axis in a region with a more complicated dispersion pattern. Modes of different overtone numbers seem to join into new branches of more or less constant slopes corresponding to the P-wave group velocity.

It can be seen in Figs 1, 3 and 5 that an isolation filter, \( f^0(t) \), resembles the synthetic seismogram near the arrival time of the target wave group, and the two waveforms are very different elsewhere. It can be expected that the real-Earth isolation filter, \( f(t) \), is a distorted version of the computed one, \( f^0(t) \), for the reference model. Assuming a sufficient knowledge on both the source mechanism and the instrument response, the discrepancies in the amplitudes and phases of the two isolation filters \( f^0(t) \) and \( f(t) \) are caused solely by the deficiencies of the reference model and would have been useful data if they could be linearly inverted. Gee & Jordan (1992) showed that within a narrow frequency band, the difference between the two isolation filters can be characterized by a five-parameter Gaussian wavelet. These parameters, obtainable by fitting a narrow-band cross-correlagram between \( f^0(t) \) and \( f(t) \), are the most applicable kind and on both the source mechanism and the instrument response, the differential phase delays, \( \delta_i \), are the most amendable kind and will be the focus of attention in this study.

The GSDF measurements obtained from (2) cannot be directly inverted for two reasons. First, the cross-correlagram between \( f^0(t) \) and \( s(t) \) includes not only the effect of real-Earth structure on the target wave but also its interference with other waves in \( s(t) \) having similar arrival times and yet not modelled in \( f^0(t) \) (for example the ScS wave and other weaker arrivals in Fig. 3) as well as the structural effect on those interfering waves. This interference contribution can be corrected by taking into account the cross-correlation between \( f^0(t) \) and the full synthetic, namely subtracting from the raw GSDFs the corresponding measurements made on the time series in (2) with \( s(t) \) replaced by \( s^0(t) \); subsequently the sensitivity kernel is also adjusted to account for the effect of the structural perturbations on the unmodelled waves. Second, although the measurements are conducted around the centroid frequencies \( \omega_i \), they do not reflect the propagation properties of waves with the same frequencies since each of the windowing...
Figure 4. Toroidal-mode dispersion diagrams for earth model PA5 showing the weighting factors in the partial normal-mode summations for the respective isolation filters (a) and (b) in Fig. 3. For both diagrams, the sizes of the open circles are proportional to the values of the weighting factors for the corresponding modes. The straight diagonal lines correspond to the ray parameter values at which the $S$ wave grazes, counterclockwise from the right, the Moho, the 410-km and 660-km discontinuities, and the CMB. The branch numbers $n = 0, 10, 20, 30$ and 40 are indicated at the top of each diagram.

Figure 5. Vertical-component displacement seismogram at station HON from a shallow-focus event in Fiji. The synthetic is computed for model PA5 by normal-mode summation. Trace (a) is the isolation filter for the direct $P$ wave obtained by weighted normal-mode summations.
and filtering operations causes a shift in the central frequency as well as a change in the band width. Therefore, another adjustment is necessary to account for the effects of the time- and frequency-domain localizations; the explicit equations are given by Gee & Jordan (1992, eqs 56–59). The corrected GSDF measurements, for example the differential phase delays, $\delta \tau$, are the final data. The right panels of Figs 1 and 3 display some of the frequency-dependent phase-delay measurements.

To avoid misinterpreting the measurements, further care must be taken to ensure that their sensitivities to the model structures are correctly represented by the kernels used in the inversion. In the case of differential phase delays, since they can be expressed as linear functionals of the eigenfrequency shifts, $\delta \omega_j$, of individual modes through the Gaussian-wavelet summation theorem (Gee & Jordan 1992, Appendix E), the kernels can be obtained through the linear relationship between the eigenfrequency shifts and structural perturbations provided by the first-order normal-mode perturbation theory (Woodhouse & Dahlen 1978). The kernel consistent with a specific measurement is calculated with the same set of normal modes and their weighting factors used in creating the isolation filter. It is then corrected for the effect of model perturbations on waves unmodelled in the isolation filter. This is done by including modes contributing to the unmodelled waves. As a result of this correction, the final kernel is computed effectively by a new and usually larger set of normal modes with different weighting factors that corresponds in fact to both the modelled and unmodelled waves (e.g. the Love + ScS and other waves in the case of Fig. 3). Although alternatively one could start with an isolation filter with the larger mode set for the Love + ScS and other possible interfering wave groups, which renders the interference corrections to the GSDF measurements and the kernels unnecessary, the correction mechanism in the GSDF analysis is much more appealing because it retrieves the information corresponding to a simpler, and thus more physically interpretable, isolation filter. However, the two procedures are equivalent (see examples in Figs 5 and 6 of Gaherty et al. 1996).

The inversion of the frequency-dependent differential phase delays provides an alternative tomographic scheme in which the data and the kernels are mutually consistent, and they reflect to a great extent only the influence of the Earth’s structure. It is more robust than the conventional traveltime tomography since no geometrical optics approximation is made, the measurements are more objective and reliable, and the structural effect is more realistically represented. It is a waveform-driven method. However, the structural dependence of the waveform is not linearized but merely shifted to its nonlinear functionals: the narrow-band measurements which are themselves more linearly dependent on the model perturbations. The resulting inversion, therefore, does not require a particularly good reference model to start with because, as illustrated by eq. (88) in Gee & Jordan (1992), a frequency-independent time shift of the entire target wave group does not contribute to the final data. Moreover, the fact that the narrow-band measurement can be made at different frequencies provides a greater number of constraints on the model and leads to better resolving power in the tomographic inversion.

3 SENSITIVITY KERNELS IN AN ASPHERICAL EARTH

Our purpose is to extend the GSDF methodology discussed in the previous section so that lateral heterogeneities can be taken into account properly and 3-D Earth models can be inverted. In this section, we derive the expressions, with the help of normal-mode coupling theory (e.g. Woodhouse 1980, 1983; Li & Tanimoto 1993), for the 3-D Fréchet kernels for all the
model parameters. Then the 2-D kernels within the source-receiver great-circle plane are obtained via a stationary-phase approximation.

3.1 3-D Fréchet kernels of traveltime residuals

The weighted normal-mode summation (1) for the isolation filter can also be written in a matrix form:

$$f^0(t) = \mathbf{R}^T \cdot \text{EXP}(-i\mathbf{\Omega}^0 t) \cdot \mathbf{S},$$  

(4)

where superscript \( T \) stands for transpose and the weighting factors \( \omega_i \) in (1) have been absorbed into the vector \( \mathbf{R} \). The diagonal matrix \( \mathbf{\Omega}^0 \) in the exponent contains the degenerate eigenfrequencies of the normal modes in the reference model:

$$\mathbf{\Omega}^0 = \begin{pmatrix}
\nu_1 & 0 \\
0 & \nu_2 \\
& \ddots \\
0 & & & \nu_N
\end{pmatrix},$$  

(5)

and the exponential of a matrix \( \mathbf{A} \) is defined through the Taylor series

$$\text{EXP}(\mathbf{A}) = \mathbf{I} + \frac{1}{2} \mathbf{A} \cdot \mathbf{A} + \cdots + \frac{1}{n!} \mathbf{A}^n + \cdots,$$  

(6)

with \( \mathbf{I} \) the identity matrix. Here we use the bold-faced uppercase \( \text{EXP} \) to indicate that the resulting quantity is a matrix.

Based upon the first-order normal-mode perturbation theory, the seismogram in a slightly perturbed Earth model, \( s(t) \), has the matrix form expression (Woodhouse 1983; Li & Tanimoto 1993)

$$s(t) = \mathbf{R}^T \cdot \text{EXP}(-i\mathbf{\Omega} \cdot t) \cdot \mathbf{S},$$  

(7)

where \( \mathbf{R} \) and \( \mathbf{S} \) are the vectors whose elements are the receiving and source excitation functions similar to those in (1). Their dimensions here, however, are generally infinite as opposed to only \( N \) in (1) and (4) for the isolation filter. The dimensions can be made finite if a certain normal-mode coupling scheme is adopted so that each mode is coupled with only a limited number of other modes. In our procedure, we utilize the weighting scheme for obtaining the isolation filter to specify the significant mode-coupling effect. In (7), the matrix \( \mathbf{\Omega} \) in the exponent is non-diagonal and can be written as

$$\mathbf{\Omega} = \mathbf{\Omega}^0 + \delta \mathbf{\Omega},$$  

(8)

where \( \mathbf{\Omega}^0 \) is the diagonal matrix of the degenerate eigenfrequencies similar to the one in (5) and the elements of the non-diagonal matrix \( \delta \mathbf{\Omega} \) are

$$\delta \mathbf{\Omega}_{ij} = \frac{\delta \mathbf{Z}_{ij}}{\omega_i + \omega_j},$$  

(9)

where \( \omega_i \) and \( \omega_j \) are the degenerate real eigenfrequencies of the \( i \)th and \( j \)th mode, respectively, and \( \delta \mathbf{Z}_{ij} \) are the elements of the normal-mode coupling matrix \( \delta \mathbf{Z} \) whose expressions have been provided by Woodhouse (1980). We use \( \delta \mathbf{Z} \) instead of the usual \( \mathbf{Z} \) to emphasize that its elements are all quantities of the first order in model parameter perturbations.

In (7), it is obvious that the seismogram \( s(t) \) based on normal-mode summation is related to the model perturbations in a non-linear fashion, which limits the direct use of waveforms in tomographic inversions. Some linearized functionals of structural perturbations, such as the shifts of normal-mode eigenfrequencies or their splitting functions, can be constructed from the seismograms through rather sophisticated data processing procedures and inverted for spherical and aspherical earth models (e.g. Gilbert & Dziewonski 1975; Giardini, Li & Woodhouse 1987; Widmer & Masters 1992). The application of these functionals, however, is somewhat limited since they either tend to be sensitive much more strongly to the even degrees of the lateral heterogeneity or become more difficult to use at relatively high frequencies. Therefore, seismic tomographies based on normal-mode theory and with data measured from waveforms, such as the seismograms themselves or phase perturbations, are generally non-linear (Woodhouse & Dziewonski 1984). Two types of compromises are usually made to render the inversion schemes more feasible. One is to retain the non-linearity in the inversion of each single seismogram and reduce the computational effort required in the non-linear optimization by keeping the number of effective model parameters to the minimum (e.g. \(< 10 \) (Nolet 1990; Marquering & Snieder 1995). The other one is to linearize directly the expression in (7) with the first-order Born approximation under the assumption (Woodhouse 1983)

$$\| \delta \mathbf{\Omega} \| \ll \| \mathbf{\Omega}^0 \|,$$  

(10)

where \( \| \cdot \| \) represents an appropriate matrix norm. This approximation has been used in the computation of long-period synthetic seismograms in anisotropic aspherical earth models (Su, Park & Yu 1993; Park 1997) and the inversion of lateral heterogeneities of shear-wave velocity in the mantle (Li & Romanowicz 1995, 1996). We take the opportunity to point out that the condition under which the first-order Born approximation to (7) is valid has sometimes been stated as \( \| \delta \mathbf{Z} \| \) to be small, where \( \tau \) is the time after the earthquake origin time (e.g. Li & Romanowicz 1995). The true criterion in (10) for linearizing (7), however, requires effectively that the normal-mode eigenfrequency shift caused by lateral heterogeneity be significantly smaller than the corresponding degenerate eigenfrequencies, which is much less restrictive than the measure of \( \| \delta \mathbf{Z} \| \). However, because of the presence of degeneracy, the first-order Born approximation produces a secular term whose amplitude grows linearly with time. This imposes another criterion, not to be confused with that in (10), for the first-order Born approximation to be physically acceptable. A comparison between synthetic seismograms obtained with the first-order Born approximation and Galerkin method indicates that for an anisotropic aspherical model of degree 6 with about 2 per cent perturbation, the first-order Born approximation provides sufficient accuracy for up to 5 hr after the origin time (Su et al. 1993). In Li & Romanowicz (1995, 1996), a path-averaged non-linear inversion is done for each of the source–station pairs before carrying out the global linear inversion.

In this study, we apply the first-order Born approximation in deriving the time-domain expressions for the Fréchet kernels of the frequency-dependent traveltime residuals for the reason of a well-established mathematical framework. Equivalent expressions can also be obtained through a more laborious procedure in the frequency domain in which the secular term can be avoided. Therefore, it is important to point out that,
because of the ability to handle any constant time shift of the target wave group mentioned in Section 2, these kernels can still be used in the GSDF inversion as long as the criterion in eq. (10) is valid. The linearized version of (7) is of the form (Woodhouse 1983)

\[ s(t) = \hat{s}(t) + \delta s(t), \] (11)

where

\[ \hat{s}(t) = R^T \cdot \text{EXP}(-i\Omega t) \cdot S = \sum_j R_j S_j \exp(-iy_j t), \] (12)

is the complete seismogram in the reference model, and

\[ \delta s(t) = \frac{1}{\omega_0^2 - \omega_0^2} \sum_j R_j \delta Z_{ij} S_j, \] (13)

is the waveform perturbation which is linearly related to the model perturbations by virtue of \( \delta Z_{ij} \), and therefore can be written symbolically as

\[ \delta s(t) = \int_0^t \mathbf{U}^T(t; \tau; r; r_s) \left[ \frac{\partial \mathbf{m}(r)}{\partial \mathbf{m}_0(r)} \right] d^3r, \] (14)

where \( r_s \) and \( r \) are the source and receiver locations, respectively, \( \left[ \frac{\partial \mathbf{m}(r)}{\partial \mathbf{m}_0(r)} \right] \) is a symbolic notation for a vector whose elements represent the perturbations of the model parameters relative to the reference model, for example \( (\delta \rho/\rho_0) \), at the location \( r \) (see eq. B6), and the elements of the vector \( \mathbf{U} \) are the corresponding Fréchet kernels. Throughout this paper, subscripts \( S \) and \( R \) represent the source and receiver, respectively. Furthermore, we drop the subscript \( 0 \) in the individual elements of the symbolic vector of the model perturbations, for example we use \( (\delta \rho/\rho) \) instead of \( (\delta \rho/\rho_0) \) for the density perturbation. The integral in (14) is over the entire volume of the Earth \( \Omega \).

Substituting (1), (3) and (11) into (2) and applying the condition for the apparent differential phase-delay measurements

\[ \left\{ \frac{d}{dt} \left[ F_i W C_p(t) \right] \right\} = 0, \] (15)

we obtain the expression for the narrow-band time delays:

\[ \delta t_i = \frac{1}{P_i} \int_{t_i}^{t_f} j_0^0(\tau) \delta s_0(\tau) d\tau + \tau_0^0, \] (16)

with

\[ P_i = \frac{1}{\int_{t_i}^{t_f} j_0^0(\tau) s_0(\tau) d\tau}, \] (17)

and

\[ \tau_0^0 = \frac{1}{P_i} \int_{t_i}^{t_f} j_0^0(\tau) s_0(\tau) d\tau, \] (18)

where \( j_0^0(\tau) \) and \( j_0^0(\tau) \) are respectively the first and second derivatives of the isolation filter with respect to time and the integration limits in (16)–(18) have been changed to the two ends of the time window \( W(t) \). It is worth pointing out that, in applying (15), we have assumed that the phase delay is measured by the time lag at which the narrow-band cross-correlagram achieves its maximum. In the original GSDF measurement, however, this quantity is one of five parameters determined by fitting a Gaussian wavelet through a non-linear optimization. The quantities \( s_0^0(\tau) \) and \( \delta s_0(\tau) \) are the waveform and its perturbation after the narrow-band filters \( F_i \) are applied. It is noteworthy that \( s_0^0 \) in (18) is the frequency-dependent phase delay measured by the narrow-band cross-correlagrams between the isolation filter and the full synthetic, which is the necessary interference correction for the phase-delay measurements caused by the waves not properly modelled in the partial normal-mode summation of the isolation filter. It can be shown that \( s_0^0 \) vanishes if the isolation filter \( f(\tau) \) is identical to \( s_0^0(\tau) \) in the time interval \([t_i, t_f]\).

The first term on the right-hand side of (16) provides us with the Fréchet kernels of the time delays:

\[ K_i(r; r_s; r_k) = \frac{1}{P_i} \int_{t_i}^{t_f} j_0(\tau) U_i(r; \tau; r_s; r_k) d\tau. \] (19)

In Appendix B, it is shown that the normal-mode coupling matrix \( \delta Z_{ij} \) can always be written in the form (see eq. B5)

\[ \delta Z_{ij} = \int_0^1 \left[ \sum_{M = -2}^2 A_{M}^{ij}(r) Y_{M}^{*} \left( 0, \phi \right) Y_{M} \left( 0, \phi \right) \right] \left[ \frac{\partial \mathbf{m}(r)}{\partial \mathbf{m}_0(r)} \right] d^3r, \] (20)

where \( A_{M}^{ij}(r) \), discussed in detail in Appendix B, is a vector each of whose elements contains the radial part of the Fréchet kernel for one particular type of model parameter, such as density or an elastic constant. The functions \( Y_{M} \left( 0, \phi \right) \) are the generalized spherical harmonics (GSH) introduced in Pinney & Burridge (1973). We have added an extra factor of \( 2l + 1/4\pi \) so that \( Y_{M} \equiv Y_{M} \), the fully normalized ordinary spherical harmonics (Edmonds 1960). The superscript * indicates complex conjugation. Collecting all the expressions for \( R_i \), \( S_j \) and \( \delta Z_{ij} \), we have

\[ U_i(r; \tau; r_k; r_s) = \sum_{M = -2}^2 \frac{2}{4\pi} F_{ik} F_{ik} \exp(-i\omega_k t) - \exp(-i\omega_k t) A_{M}^{ij}(r) \]

\[ \times I_i(r_k) E_i(r_k) \left[ \sum_{l = -1}^2 Y_{l} \left( 0, \phi_s \right) Y_{l}^{*} \left( 0, \phi \right) \right] \]

\[ \times \sum_{m = -2}^2 Y_{M}^{*} \left( 0, \phi_s \right) Y_{M} \left( 0, \phi \right), \] (21)

where \( F_{ik} = F_i(\omega_k) \), \( I_i(r_k) \) and \( E_i(r_k) \) are operators derived from \( R_i \) and \( S_j \), respectively, whose explicit expressions are given in Appendix A. Using the addition theorem of the GSH (Edmonds 1960) to carry out the summations over the azimuthal orders \( m \) and \( m' \), we obtain

\[ U_i(r; \tau; r_k; r_s) = \sum_{M = 2}^2 \sqrt{\frac{2l + 1}{4\pi}} \sqrt{\frac{2l' + 1}{4\pi}} F_{ik} \]

\[ \times \exp(-i\omega_k t) - \exp(-i\omega_k t) A_{M}^{ij}(r) \]

\[ \times I_i(r_k) E_i(r_k) \left[ (M_0 \delta_{M_0} Q_0 - M_0 \delta_{Q_0}) \right] \]

\[ \times X_M(\theta \phi) Y_M(\theta \phi), \] (22)

The definitions of the angles in the arguments of the spherical harmonics are shown in Fig. 7. The function \( Y_{M} \left( 0, \phi \right) \) is the real amplitude of \( Y_{M} \left( 0, \phi \right) \), i.e.

\[ Y_{M} \left( 0, \phi \right) = X_M(\theta \phi) \exp(i\phi), \] (23)
of the earth. A widely employed way of reducing the computational labour is to focus the attention only on the contribution from lateral structure within the source–receiver great-circle plane. Li & Tanimoto (1993) showed how to achieve this by partially evaluating the volume integral in (14) on a sphere of unit radius in the direction perpendicular to the source–receiver great-circle plane with the method of stationary phase under the assumption that the lateral perturbations of the model parameters are smooth. Without loss of generality, we can rotate the coordinate system for a given pair of $r_s$ and $r_h$, with epicentral distance $\Delta$, such that the source and receiver are both in the equatorial plane and that they are on the meridians of 0 and $\lambda$ longitudes, respectively. The resulting 2-D version of $K_\alpha(\mathbf{r}; r_s; r_h)$ at $\tilde{\mathbf{r}} = (r, \varphi)$ in the source–receiver great-circle plane (the equatorial plane in the rotated coordinate system) is of the form

$$K_\alpha(\mathbf{r}; r_s; r_h) = \frac{F_\alpha}{\pi} \frac{\cos(\alpha_2 \tau) - \cos(\alpha_1 \tau)}{\cos^2(\tau/2) - \cos^2(\tau/2)} \mathcal{A}^{\dagger}_\alpha(r) \mathcal{L}_s(r_h) \mathcal{E}_d(r_h)$$

which relates the time-delays $\delta t_i$ to the model perturbations by

$$\delta t_i = \delta t_i - \tau_i^0 = \int_0^l K_\alpha^\dagger(\mathbf{r}; r_s; r_h) \cdot \frac{\partial m(\mathbf{r})}{\partial m_0(\mathbf{r})} \ d\mathbf{r}.$$  

Here we have arrived at the linear relationship between the frequency-dependent traveltime residuals $\delta t_i$ and the model perturbations.

### 3.2 2-D Fréchet kernels of traveltime residuals

The expression in (24) provides the kernels for the properly corrected narrow-band phase-delay measurements, $\delta t_i$, and can therefore be applied in linear inversions of 3-D earth models through the relation in (25). However, the calculation involved in (24) is impractically heavy for any inversion scheme with currently available computational facilities because of the double summations over $k$ and $k'$ as well as the need to evaluate the kernels at all grid points over the entire volume

$$D = \left(l + \frac{1}{2}\right) \cos \varphi \sin(\Delta - \varphi) + \left(l + \frac{1}{2}\right) \sin \varphi \cos(\Delta - \varphi).$$

The elements of $\mathcal{A}^{\dagger}_\alpha(r)$ are combinations of the real parts of $\mathcal{A}^{\dagger}_\alpha(r)$, i.e.

$$\mathcal{A}^{\dagger}_\alpha(r) = \sum_{M=\pm 2} \mathcal{R}_e[\mathcal{A}^{\dagger}_\alpha(r)].$$

As can be seen from the expressions of $\mathcal{A}^{\dagger}_\alpha(r)$ in Appendix B for the perturbations in elastic tensor elements, density and topographies of discontinuities, the spheroidal–toroidal coupling terms appear only in the imaginary parts of $\mathcal{A}^{\dagger}_\alpha(r)$. Therefore, for the approximations discussed in this paper, the spheroidal and toroidal modes are uncoupled.

The coefficients $C_{\lambda\lambda}$ and $S_{\lambda\lambda}$ in (26) are related to the moment tensor, the instrument response as well as the normal-mode eigenfunctions at the source and receiver locations. Their expressions are provided in Appendix C. With the 2-D kernel in (26), the linear relation (25) becomes

$$\delta t_i = \int_0^\Delta K_\alpha^\dagger(\mathbf{r}; r_s; r_h) \cdot \frac{\partial m(\mathbf{r})}{\partial m_0(\mathbf{r})} \ d\mathbf{r}.$$

where $\alpha$ is the radius of the Earth. Eq. (29) provides the linear relationship between the traveltime residuals $\delta t_i$ and the model perturbations $[\partial m(\mathbf{r})/\partial m_0(\mathbf{r})]$ within the source–receiver great-circle plane, which forms the basis of the 2-D tomographic inversion.
4 EXAMPLES OF 2-D SENSITIVITY KERNELS

We have seen in Section 3 that it is computationally more practical to relate the GSDF measurements to the laterally heterogeneous perturbations of the model parameters within the source-receiver great-circle plane. With the linear relation in (29) and the 2-D Fréchet kernels in (26), it is possible to invert for 2-D earth models with observations of various target groups of waves at a single station from multiple events in a single seismic region. The resolving power is enhanced by utilizing events at various depths and measuring the traveltime residuals or differential phase delays at different frequencies. We have implemented this procedure and conducted a 2-D tomographic inversion for the upper-mantle anisotropic lateral structure along the central Pacific corridor between Tonga and Hawaii based on the path-averaged transversely isotropic model PA5 (Gaherty et al. 1996). Issues related to the details of the measurement and inversion of the frequency-dependent traveltime residuals are discussed in Katzman et al. (1998). In this paper, we focus on the frequency-dependent 2-D kernels \( K_2(\mathbf{r}, \mathbf{r}_0, \mathbf{r}_1) \) and investigate their characteristics through various numerical examples.

Following the procedure discussed in Section 2 for the GSDF analysis, we first create an isolation filter for a chosen target wave group by a weighted normal-mode summation. The number of modes as well as their weights in the summation are determined based on a number of criteria, such as the overall frequency band, group- and phase-velocity (or ray parameter) windows and energy partition, so as to model as closely as possible the waveform in the time window centred around the arrival time of the target wave and suppress all the other waves. Since the 2-D kernel expressions contain a time integral (see eq. 26), the amount of computation involved is directly linked to the length of the time window considered in the computation of the cross-correlagram. The windows are usually wider for surface waves than for body waves. For intermediate and deep events, the direct-wave arrivals (such as \( S \)) and the corresponding near-source surface reflected ones (such as \( S_d \)) arrive at about the same time with similar phase velocity and are sometimes difficult to isolate from each other. A wider time window is therefore necessary to include the entire target wave train. In practice, we have used time windows of varying lengths ranging from 100 to 350 \( s \) according to the measuring frequency and the purity of the target wave group. In order to reduce the computational effort further, we have also tried to limit the number of modes involved in each isolation filter by selecting as target the wave groups dominated by constituents having similar ray parameters and have ignored the corrections to their kernels for the effect of model perturbations to the minor arrivals unmodelled in the isolation filter. For example, in the Love-wave isolation filter in Fig. 3, the \( ScS \) and other possible interfering waves arriving within the same Love-wave time window are not modelled since they are from modes of much smaller ray parameters and also are much weaker. Therefore, the kernels are computed with only the modes and weighting factors in Fig 4(a) for the Love wave. The reason for this is that, as explained in Section 2, the kernel corrections effectively utilize a larger normal-mode set and therefore are not efficient when mode coupling is considered. However, corrections to measurements due to the interference of minor arrivals, represented by \( \tau^n_0 \) in (16), are not ignored since they only account for the propagation of the unmodelled waves in the reference model and therefore do not involve mode-coupling computation. With this consideration, the number of modes is brought down to around 1300 for most of the isolation filters in both toroidal and spheroidal cases (as shown in Figs 2, 4, and 6). Also as a result of this, for observations from the Tonga–Hawaii path with epicentral distances of 38°–57°, a number of the transverse-component \( SS \) phases have to be excluded due to the possible significant interference with the \( ScS \) waves having similar arrival times but very different ray parameters. After the creation of the isolation filter, the same set of normal modes is used in the computation of its 2-D kernels. In this normal-mode subset, we adopt a coupling scheme in which any two modes are coupled if the differences in their eigenfrequencies and group velocities are less than 10 mHz and 2 km s\(^{-1}\), respectively, in order to reduce further the amount of computation. The accuracy of the 2-D kernels obtained from this mode-coupling scheme can be assessed by comparing them with those obtained from complete mode-coupling computations. We have found this coupling scheme to be both efficient and sufficiently accurate.

4.1 The effect of normal-mode coupling on the 2-D kernels

The effect of normal-mode coupling due to lateral heterogeneity has been the subject of study in a number of publications (Park 1987; Romanowicz 1987; Li & Tanimoto 1993). It is usually the most important factor in determining the geometrical shapes of the Fréchet kernels. To examine the effect of different normal-mode coupling considerations on the 2-D kernels of the traveltime residuals, we choose the Love wave \( G1 \) and the direct \( SH \) wave in Fig. 3 as target wave groups. It can be seen that for the direct \( SH \) and Love waves, the transversely isotropic reference model PA5 provides a fairly good waveform fit between the synthetic and the data.

In Fig. 8, the Fréchet kernels of the Love wave \( G1 \) for the centroid frequency of 25 mHz under various coupling considerations are plotted. These are sensitivities of the Love-wave traveltime residuals (the triangles in the right panel of Fig. 3) to the isotropic \( S \)-wave velocity, which is defined as the average of the two \( S \)-wave velocities, i.e. \( \beta = (\beta_s + \beta_h)/2 \), where \( \beta_s \) and \( \beta_h \) are the velocities for the horizontally propagating \( SV \) and \( SH \) waves, respectively, and \( \beta_s \) is also the velocity of the vertically propagating shear waves. In Fig. 8, the radial dimensions in all the plots are exaggerated by a factor of 2 in order to display the details of the kernels. All the shaded regions in these plots have negative amplitudes indicating that a positive perturbation in \( \beta_s \), or a velocity increase, causes a negative traveltime residual, or a traveltime advance. For the kernel in Fig 8(a), there is no consideration of inter-multiplet coupling and, as expected, the resulting kernel is only radially dependent and uniform in lateral direction around the entire concentric circle. It is obvious that with single source-receiver path coverage, such a kernel cannot provide any resolution of the lateral structure. When the coupling is considered between modes with the same overtone number \( n \), often referred to in the literature as along-branch coupling, the amplitude outside the source-receiver minor arc vanishes, leading to the commonly presumed sensitivity for the \( G1 \), as depicted in Fig 8(b). This type of quasi-1-D kernel, laterally uniform between the...
source and receiver, has been adopted in almost all long-period waveform tomographic studies. Any lateral resolution with this coupling consideration can only be achieved by using a variety of different source-receiver pairs either in the same great-circle plane (Nolet 1990) or in many crossing great-circle planes (e.g. Zielhuis & Nolet 1994). However, when coupling is considered between normal modes of different branches, the sensitivity becomes non-uniform in lateral direction between the source and receiver and thus provides lateral sensitivity, as can be seen in Fig. 8(d). In addition, the Fréchet kernel in this case takes on the shape of an SSS(H) wave path. This is because at $\Delta = 44^\circ$, the SS wave is simply part of the Love wave group and becomes dominant at 25 mHz. One can see here that the distinction between surface and body waves is not clear-cut. A similar mode-coupling effect can be seen in Fig. 9 for the Fréchet kernels of the direct SH wave for a centroid frequency of 30 mHz. Like those for the Love wave, the amplitudes of the kernels in Fig. 9 are negative. From the very different geometrical features of the 2-D kernels, we can draw the conclusion that normal-mode coupling clearly improves the resolution achievable in the inversion of the traveltime residuals.

In addition to introducing lateral resolving capability, the normal-mode coupling also provides more realistic radial sensitivity. This can be observed by comparing the radial sensitivities of the kernels obtained with the three different coupling considerations. The radial sensitivity is simply the integrand in (29) after carrying out the $\varphi$-integration. In Fig. 10, the three types of radial sensitivities to $V_\beta$ are plotted for a vertical-component SSS wave from an event with an epicentral distance of 45° and a focal depth of 419 km. Clearly, the three coupling schemes yield the same radial functions. This is because, for the SSS wave, the normal modes contributing to its isolation filter are all from the lowest few branches which are very smooth, similar to those for the Love wave in Fig. 4. The radial eigenfunctions of these modes are similar to one another so that different coupling computations do not cause any significant change in the lateral part of the kernels, but only introduce the difference in the lateral dependence. Here we should also point out that these radially dependent kernels, especially the uncoupled ones, are qualitatively the same as those in Fig. 8(c) of Geherty et al. (1996) computed without coupling consideration for the same SSS wave from the same event, which affirms our confidence in the 2-D kernels. There are, however, some differences because the kernels in Geherty et al. (1996) were calculated for a different isotropic reference model PA2 (Lerner-Lam & Jordan 1987) with a slightly different set of normal modes and weighting factors, and the expressions for the 2-D kernels were derived with a different approach.

In Fig. 11, the same comparison is made for a vertical-component S wave from an event with an epicentral distance of 50° and a source depth of 575 km. For the direct S wave, the contributing modes belong to many different branches (as seen in Fig. 2) and their radial eigenfunctions can be quite different. Therefore, radial sensitivity considering full normal-mode coupling is very different from the other cases. In particular, the fully coupled radial kernel seems to strengthen the sensitivity near the turning depth of the direct S wave, resulting in a more realistic description for the higher concentration of waves in that region. It is also interesting to see that the along-branch coupling does not make much difference in the radial sensitivity as compared with the uncoupled case. The reason is that, along the individual branches, the modes with very different group velocities, and thus different radial eigenfunctions, are excluded (the breaks in the branches in Fig. 2). The effect of the along-branch coupling is simply that of taking the uncoupled kernels, such as the ones in Figs 8(a) and 9(a), and folding the sensitivity beyond the source and receiver onto the region between them. In the remainder of this paper, all the 2-D kernels are computed with inter-multiplet normal-mode coupling.

### 4.2 Fundamental and overtone modes in surface waves

Surface waves have played a very important role in most waveform tomography studies with the understanding that they are sensitive to only the averaged and depth-dependent structure between the source and the receiver. In this context, either the coupling between modes of different branches is...
mantle in the SS(H) body-wave fashion, as shown in Fig. 8(d). Therefore, inverting the measurements from G1 with only depth-dependent kernels may lead to bias since any localized structure is averaged out and at the same time deeper heterogeneities are projected to shallower depth. From this observation, one can expect that the 2-D Fréchet kernels obtained with the inclusion of all contributing modes and the consideration of mode coupling will lead to better tomographic models with higher resolving power.

Unlike the Love waves, the Rayleigh-wave phases are dominated by the fundamental modes. This can be understood from the difference between the fundamental-branch spheroidal modes and those in other branches, as seen in Figs 2 and 6. In the seismograms of the Kermadec event in Fig. 1, the top trace is the synthetic for the Rayleigh wave obtained with only fundamental modes. It is almost identical to the isolation filter in trace (a) for which overtone branches are included. The sensitivity kernels corresponding to the two Rayleigh-wave partial synthetics are plotted in Fig. 12. Although the inclusion of higher overtone modes in Fig. 12(b) does introduce some SSS-wave sensitivity with deeper penetration, the difference between these two kernels is much less pronounced than that between the kernels in (c) and (d) in Fig. 8 for the Love wave.

### 4.3 Frequency dependence of the 2-D kernels

One of the distinctive features of the GSDF method is its practice of conducting measurements at multiple frequencies. The windowing and narrow-band filtering that inevitably change both the centroid frequencies and their bandwidths are accounted for in the GSDF process to ensure the consistency in the inversion between the measurement and Fréchet kernels. The frequency-dependent measurements as well as their sensitivity kernels greatly enhance the information content that can be extracted from each individual seismogram and the resolving power of the tomographic inversion.

The strong variation of the traveltime residuals with frequency can be seen clearly in Figs 1 and 3. The non-monotonic frequency dependence is an indication of the complicated frequency-dependent influence of the model perturbations on the propagation of these seismic waves. Obtained from coupled-mode summation, the 2-D kernels are the results of the interference of the contributing normal modes and their scattering derivatives due to the perturbations. As the frequency changes, the properties of the normal modes, such as the shapes of their eigenfunctions and the partitions between the compressional and shear energies in spheroidal modes, will also change. It has been demonstrated through the correspondence between normal modes and seismic rays that higher frequency often leads to more types of interfering seismic waves (such as S and ScS in the toroidal case or P, S, PS, ScS and SKS in the spheroidal case) as well as more accurate descriptions of normal modes by seismic rays, due to a better mode–ray correspondence (Zhao & Dahlen 1993). The complicated variations of the interference patterns of various waves will be discussed in Section 4.5. Here we only look at the simplest and most direct frequency effect through the 2-D kernels for the Love wave in Fig. 3 and the vertical-component SS wave in Fig. 1. In Fig 13(a), the kernel for the Love wave at a centroid frequency of 15 mHz is plotted. One can see that at this lower frequency the kernel shows a weaker lateral variation than its counterpart at the higher frequency of 35 mHz in
Figure 10. Radially dependent sensitivity kernels of the vertical-component SSS wave to $\beta_v$ around the centroid frequencies of 15 and 40 mHz. The event has an epicentral distance of 45° and a focal depth of 419 km. These radial functions are obtained by integrating the types of 2-D kernels in Figs 8 and 9 horizontally over the entire circle at each level. In order to compare with Fig. 8(c) in Gaherty et al. (1996), conversion to the 1-D kernels is done so that the unit in the horizontal axis is $10^{-8}$ s km$^{-3}$.

Figure 11. Radially dependent sensitivity kernels of the vertical-component $S$ wave to $\beta_v$ around the centroid frequencies of 15 and 40 mHz. The event has an epicentral distance of 50° and a focal depth of 575 km. These radial functions are obtained by integrating the types of 2-D kernels in Figs 8 and 9 horizontally around the entire circle at each level. In order to compare with Fig. 9(c) in Gaherty et al. (1996), conversion to the 1-D kernels is done so that the unit in the horizontal axis is $10^{-9}$ s km$^{-3}$.

Fig. 13(b), which confirms the notion that for long-period surface waves it is more justified to assume path-averaged and only depth-dependent sensitivity kernels. However, it is also clear that this assumption has lost its validity slightly, even at this low frequency of 15 mHz. Fréchet kernels of higher frequencies, as shown in Fig. 13(b), define clearer ray paths, a direct consequence of the better mode-ray correspondence.

In Figs 14(a) and (b), the Fréchet kernel of the vertical-component $SS$ wave in Fig. 1(b) is plotted for centroid frequencies of 20 and 40 mHz, respectively. Negative and positive amplitudes are shown by the orange and blue colours, respectively. Once again the sensitivity spans a broader region at lower frequency but is more concentrated around the multiple geometrical ray paths at higher frequency. As indicated
4.4. 2-D Fréchet kernels for radially anisotropic perturbations

Gaherty et al. (1996) discussed the necessity of explaining the GSDF measurements for the Tonga–Hawaii corridor by a radially anisotropic earth model. Our goal in extending the GSDF analysis is also to be able to deal with lateral heterogeneities of radially anisotropic structures. The striking difference in the sensitivities of the SH wave to the shear velocities and the SV and P waves to both shear and compressional velocities is illustrated by the suite of kernels in Fig. 15.

All the kernels displayed in Fig. 15 are for the traveltime residuals at the centroid frequency of 35 mHz. Panels (a) and (b) display the sensitivities of the SH wave in Fig. 3 to the shear velocities $\beta_a$ and $\beta_H$, respectively. As expected, the SH wave is more strongly influenced by $\beta_H$, the velocity of the horizontally propagating SH wave. Around the turning point, where the SH wave propagates most horizontally along its entire ray path, the sensitivity is the strongest. However, there is also a significant influence near the two ends of the SH ray path from $\beta_V$, the velocity of the vertically travelling shear wave, since away from its turning point the SH wave propagates obliquely and therefore has a vertical propagation component. In the regions near the source and receiver, the propagation is at its most vertical along its entire path due to the lower velocity at shallower depth, and the sensitivity to $\beta_V$ is the stonest.

The 2-D kernels of the vertical-component direct SV wave from the same event are plotted for the two shear velocities in panels (c) and (d) and for the two compressional velocities in panels (e) and (f). The kernel for $\beta_V$ in (c) outlines a clear $S$-wave path with negative amplitude (orange colour). The blue regions with positive amplitude as well as the apparent asymmetry in comparison with the kernels of the SH wave in (a) and (b) are the result of the interference of multiple arrivals and will be discussed in the next section. The SV wave has virtually no sensitivity to $\beta_H$. The opposite sensitivities of the SV wave to $\beta_V$ and $\beta_H$ in panels (e) and (f) can be explained in terms of the projections of its propagation and polarization vectors onto vertical and horizontal components. It is worth pointing out that the influence of the compressional velocities on the SV-wave traveltime residuals is determined by the difference in the two velocities $\zeta_V$ and $\zeta_H$, rather than their individual values. If there is no anisotropy in compressional velocity, i.e. $\zeta_V = \zeta_H = \zeta$, then a purely $S$-wave has no sensitivity to $\zeta$, as illustrated in Fig. 15(k) obtained from the simple average of the two kernels in panels (e) and (f). The resulting kernel vanishes along the $S$-wave path outlined in panel (c), except for several thin regions close to the surface. These types of sensitivities of shear waves to compressional-wave velocities have also been discussed for the 1-D case by other researchers (e.g. Dziewonski & Anderson 1981). The residual amplitudes shown in Fig. 15(k) are from the P waves in the contaminating PS and SP arrivals that are inseparable from the target $S$ wave in the isolation filter. This impurity is a consequence of the technique used in creating the isolation filter, namely the weighted normal-mode summation, and will be a common feature in the following examples of the 2-D kernels. On the other hand, if anisotropy exists in the compressional velocity, then the two kernels in panels (e) and (f) will exert different influences on the $SV$-wave propagation. This observation indicates that these $SV$ kernels for the compressional velocities can provide constraints for $\zeta_V - \zeta_H$, the measure of the anisotropy in compressional velocity, but not for the absolute values of the velocities themselves.

As another example of kernels in anisotropic models and as a confirmation of our conjecture that the residual amplitude in Fig. 15(k) is indeed from P waves, we plot in panels (g)–(j) the 2-D kernels of the vertical-component $P$ wave in Fig. 5. As for the $SV$ wave, the $P$-wave kernel for $\beta_H$ is negligible, as

earlier, negative sensitivity means that a velocity increase leads to a travelt ime advance. Positive sensitivity, on the other hand, predicts a traveltime delay for a velocity increase. This seemingly counter-intuitive phenomenon is a reflection of the fact that multiple wave packets arrive together, and is explained in Section 4.5. The two kernels in Fig. 14 have completely different sampling properties to the structure, which contributes significantly to the resolving power achieved in tomographic inversions.

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shown in panel (b). It also has little sensitivity to $b_\nu$ except in the regions close to the source and receiver, explainable by an argument similar to that for the $S'$ sensitivity to $b_m$ in panel (e). The amplitude seen in panel (g) mainly comes from various contaminating shear waves in the $P$-wave isolation filter, including primarily the $P$-to-$S$ and $S$-to-$P$ conversions at the CMB and the surface. One can see that for these waves their first legs, from the source to the CMB, are less visible than their second legs, from the CMB to the receiver. This is caused by the tunnelling effect of the $P$ wave between its turning point and the CMB and a relatively strong $P$-to-$S$ conversion at the CMB. The $P$ wave is evanescent between its turning point and the CMB and so its path is invisible. However, after the conversion at the CMB, the $S$ wave is oscillatory and its ray path can be seen clearly extending from the CMB to the surface. By comparison with the $S$-wave sensitivities to $x_\nu$ and $x_m$ in panels (e) and (f), we can conclude that the kernels emanating from the CMB in panels (i) and (j) are indeed from an $S$ wave because of their opposite signs. The blue region in (j) does not reach the surface because of a stronger negative amplitude from the $P$ wave above its turning point. Disregarding the shear-wave contributions, the $P$-wave kernel for $x_m$ in panel (j) defines a $P$-wave path, while the one for $x_\nu$ is significant only close to the source and receiver, where the propagation direction is more vertical. Panel (1) shows the average of the kernels in (i) and (j). The contributions from shear waves are effectively cancelled leading to a much clearer $P$-wave ray path. Comparing panels (k) and (l) one can conclude that the thin-skinned residual amplitude in panel (k) is indeed from the interfering $P$ waves.

4.5 Miscellaneous examples of 2D Fréchet kernels

In the examples presented so far we have examined the influence of various factors more or less separately on the 2-D sensitivity kernels of the frequency-dependent traveltime residuals. Evidently, in most circumstances these factors act together to produce very complicated characteristics in the 2-D kernels. In this section, we present a few more sensitivity kernels and seek to understand their patterns through the interference of different seismic waves and their interactions with the structural perturbations.

In Fig. 16, the 2-D sensitivities of an $SH$ wave from a deep-focus event ($h = 575$ km) are plotted for centroid frequencies of 25 mHz in panels (a) and (b) and 35 mHz in panels (c) and (d). On top of these plots, the kernels for the topographies of the 410-km and 660-km discontinuities are drawn with green curves. Since the sensitivity is generally weak above the 660-km discontinuity for such a deep event, the topography of the 410-km discontinuity has little influence on the $SH$ wave traveltime except for the kernels at 35 mHz in the region near the source. The kernel for the topography of the 660-km discontinuity is consistent with those for $b_\nu$ and $b_m$ in the sense that they have the same polarity in most places, in agreement with the idea that a depression in the 410-km or 660-km discontinuity is equivalent to a velocity decrease immediately beneath the respective discontinuity in the reference model, and vice versa. Furthermore, the comparison of the kernels for the two frequencies reveals that at higher frequencies the kernels have positive (blue region) amplitudes, as also seen in some of the previous plots, suggesting that in these regions a velocity increase leads to a more delayed arrival-time measurement.

Although it seems to be counter-intuitive at first sight, this behaviour is a result of the complicated constructive and destructive interference of the band-limited multiple seismic signals arriving within the same measuring time window. As discussed in Section 2, the partial normal-mode summation ensures that all the waves with similar group and phase velocities are included in creating the isolation filter. These different signals propagate as separate wave packets along different paths and arrive at slightly different times. Their combined sensitivity, expressed by eq. (26) in terms of coupled normal-mode summation, can also be expressed as a weighted summation of the sensitivities for individual travelling-wave packets, i.e.

$$\hat{K}_i(f) = \sum_{n=1}^{N} c_{in} \hat{K}_{in}(f),$$

where $N$ is the number of wave packets involved in the isolation filter, and $\hat{K}_{in}(f)$ is the 2-D sensitivity kernel of the $n$th packet at frequency $f$. The weighting factor $c_{in}$ is approximately (Gee & Jordan 1992, Sections 4.5 and 4.7)

$$c_{in} = C_i^{-1} a_{im} \sum_{m=1}^{N} a_{im} \exp\left[-\sigma_i(r_{im} - \tau_{im})^2/2\right] \cos[c_i(t_{im} - t_{in})],$$

(31)

where $\sigma_i$ is the half-width of the measuring frequency band and $\tau_{im}$ is the arrival time of the $n$th wave packet whose strength in the isolation filter is represented by $a_{im}$. The normalization constant in eq. (31) is

$$C_i = \sum_{m=1}^{N} \sum_{k=1}^{N} a_{im} a_{kn} \exp\left[-\sigma_i(r_{im} - r_{kn})^2/2\right] \cos[c_i(t_{im} - t_{kn})].$$

(32)

The expression in eq. (31) for the weighting factor $c_{in}$ obtained from the auto-correlogram of the isolation filter, can best be understood as an exhibition of the interference between the $n$th wave packet and the other packets. A packet of similar arrival time, i.e. $\tau_{in} \approx \tau_{im}$, interferes with it in a complicated fashion because of the phase-dependent factor $\cos[c_i(t_{im} - t_{in})]$. A packet of very different arrival time, on the other hand, has negligible interference.

For the example shown in Figs 16(a)–(d), the isolation filter is composed mainly of two seismic signals: the direct $S$ wave and the surface-reflected $S$ wave. In the case that their arrival times satisfy the relation

$$|r_{2k} - r_{1k}| \approx (2k + 1)\pi/c_{1i} < \sigma_i^{-1},$$

(33)

where $k$ is an integer, the combined sensitivity kernel becomes

$$\hat{K}_{i2}(f) = \frac{a_{1i} - a_{2i}}{a_{1i} - a_{2i}} \hat{K}_{i1}(f) - \frac{a_{2i}}{a_{1i} - a_{2i}} \hat{K}_{i2}(f).$$

(34)

Therefore, for $a_{1i} > a_{2i}$, i.e. the direct $S$ wave has a higher strength than the $S$ wave, in locations where the amplitude of $\hat{K}_{i2}(f)$ for the $S$ wave is much smaller than that of $\hat{K}_{i2}(f)$ for the $S$ wave, the combined kernel exhibits the shape of $\hat{K}_{i2}(f)$ but with a positive amplitude since the two individual kernels $\hat{K}_{i2}(f)$ and $\hat{K}_{i3}(f)$ are themselves sensitivities of travelling-wave packets and generally have a physically meaningful negative sign. This is the explanation of the blue regions in Figs 16(c) and (d). The $S$ contribution to the kernels is not so visible
after its turning point because of the dominance of the negative (orange) direct SH-wave kernels.

The 2-D kernels of the vertical-component SV wave from the same deep event for βc and the discontinuity topographies are plotted in panels (e) and (f) for centroid frequencies of 25 and 35 mHz, respectively. In panel (f), two distinct ray paths are delineated by the negative amplitudes. One of them is the direct SV wave, the target wave for which an isolation filter is created to make the measurement. The other one is the surface-reflected SP wave, which arrives a little earlier than the direct SV wave. The P–SV system obviously has more types of waves interacting with each other and therefore can be expected to have much more complex interaction patterns than the SH system. Yet the kernels in panels (e) and (f) are simple and in fact they have less polarity changes than the SH-wave kernels in panels (c) and (d). The absence of positive amplitude (blue colour) in the presence of multiple arrivals is caused by a different relation between the arrival times of the two wave packets, namely

\[ |τ_{i2} − τ_{i1}| ≈ 2kπ/ω_i < σ_i^{-1}, \tag{35} \]

which leads to the following sensitivity kernel:

\[ \hat{K}_i(γ) = \frac{a_{i1}}{a_{i1} + a_{i2}} \hat{K}_{\text{si}}(γ) + \frac{a_{i2}}{a_{i1} + a_{i2}} \hat{K}_{\text{di}}(γ). \tag{36} \]

Therefore, the combined kernel \( \hat{K}_i(γ) \) is negative everywhere. Several other phases, the S, SSP and SPPP waves, are also contained in the isolation filter and are much weaker.

The 2-D kernels in panel (g) in Fig. 16 are for the vertical-component direct SV wave in trace (c) in Fig. 1 around the centroid frequency of 40 mHz. The dominant feature in this kernel is still the region around the ray path of the direct SV wave with negative amplitude. It is enhanced by the contribution from the SS wave, which arrives almost simultaneously with the direct SV wave. The blue region with relatively large positive amplitude is the sensitivity from PS and sPS waves. It is almost as strong as the kernel for the direct SV wave. The interference between the negative (orange) and positive (blue) kernels along the second half of their ray paths results in the asymmetric shape of the kernel.

Finally, in panel (h), the kernel for vertical-component SSS waves is displayed for a centroid frequency of 40 mHz. The kernel is largely negative. The waves with positive (blue) sensitivities are all very weak.

We point out here that the relation between the arrival times of two wave packets is not necessarily always like that in eqs (33) or (35). Consequently, the combined sensitivity kernels have generally complicated shapes and polarity distributions.

5 DISCUSSION

The GSDF methodology has been extended to handle the effect of laterally heterogeneous structure in a transversely isotropic earth model and the linear relationship is established between the frequency-dependent traveltimes and the perturbations in model parameters. In this process, first-order quasi-degenerate perturbation theory is used to take the normal-mode coupling effect into account in order to achieve the realistic sensitivity of the traveltimes residuals to structural perturbations. The first-order Born approximation is utilized to derive the expressions for the Fréchet kernels. Moreover, the 2-D sensitivity kernels of the measurements are obtained with a stationary-phase integration for the inversion process to be computationally practical.

Although the complete 3-D effect of the lateral heterogeneity is approximated by its projection within the 2-D source–receiver great-circle plane, the GSDF inversion is a significant improvement over the path-average as well as ray-theoretical inversions in the sense that it describes more completely and accurately (albeit to first order) the complicated constructive and destructive interference behaviour of all types of scattered waves as well as their frequency-dependent and band-limited nature. In comparison with most of the existing techniques of waveform tomography, the linear dependence of the traveltimes on the model perturbations provides us with the possibility of handling more types of model parameters with realistic sensitivity kernels.

When the target waveform can be associated with a distinct ray-theoretical path (e.g. direct S), the 2-D kernels calculated by our technique always exhibit the strongest sensitivity along the ray path and decay with distance away from the path. In contrast, the same type of 2-D traveltime kernels obtained using ray theory and single-scattering approximation predict a pattern of sensitivity that is locally minimum along the ray path and grows with distance away from the path within the first Fresnel zone. In three dimensions, the ray-theoretical, single-scattering traveltime kernels have zero sensitivity along the ray path (e.g. Woodward 1992; Yomogida 1992). In our preliminary implementation of eq. (24) for the 3-D Fréchet kernels, however, the sensitivity kernels have patterns similar to those shown in this paper, i.e. they are maximum along ray paths. This discrepancy between traveltime kernels obtained with two different but equally well-developed approaches raises a puzzling and yet important problem to both theoretical and observational seismologists. The resolution of this problem will have strong implications in seismic traveltime tomography.

The examples of the 2-D kernels provided in Section 4 demonstrate a great variety of sensitivity of the frequency-dependent traveltime residuals to the lateral and anisotropic mantle structure. The ability of the GSDF analysis to invert consistently multi-frequency measurements from a variety of arrivals on three-component seismograms enables us not only to extract more information from each seismic record but also to interpret them in a more accurate way, with unprecedented resolving power in regional tomographic studies.

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REFERENCES


APPENDIX A: SOURCE AND RECEIVER VECTORS S AND R IN NORMAL-MODE SUMMATION

The seismogram from a point source in a spherically symmetric earth model can be expressed in terms of normal-mode summation (Gilbert 1970):

\[ D(t) = \sum_j \mathbf{v} \cdot \mathbf{e}_j(r_h) \left[ M : \mathbf{x}^*_j(r_s) \right] \exp(-i \nu_j t), \]  

(A1)

where \( \mathbf{M} \) is the moment tensor and \( \mathbf{e}_j(r_h) \) and \( \mathbf{e}_j(r_s) \) are the normal-mode eigenvectors at the receiver \( r_h \) and strain tensors at the source \( r_s \) respectively. The superscript * indicates complex conjugation. The vector \( \mathbf{v} = (v_r, v_t, v_s) \) represents the vectorial instrument response. The normal-mode eigenvectors

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and strain tensors are defined as
\[
e_j(j) = \{U_r(r)\hat{f} + V_r(r)\nabla^2 + W_r(-\hat{r} \times \nabla^2)\} Y_m(0, \phi),
\]
\[
e_k(k) = \frac{1}{2} \{(V_e + (V_e)^T)\}.
\]
(\text{A2})

where \(j = (n, l, m)\) identifies the individual normal-mode singlet and \(k = (n, l)\) indicates the corresponding multiplet, so that in a spherically symmetric earth model \(v_j \equiv v_k\). Functions \(U_r\), \(V_r\) and \(W_r\) are the radial eigenvectors of the normal mode. The operator \(\nabla^2 = \hat{h} \hat{\phi} + (\sin \theta)^{-1} \hat{\phi} \hat{h}\) represents the surface gradient on the unit sphere, and \(\hat{r}, \hat{\theta}\) and \(\hat{\phi}\) are the basis vectors of the spherical coordinate system. The orthonormal condition for the eigenvectors is
\[
\int_0^\rho \rho^2 \sin \theta \sin \phi \, d^3r = \delta_{j'j},
\]
(\text{A3})

where \(\rho\) is density and the integration is over the volume of the Earth \(\Omega\).

Woodhouse & Girmans (1982) introduced the normal-mode receiving and source excitation vectors \(\mathbf{R}\) and \(\mathbf{S}\), respectively, with the elements
\[
R_j = \mathbf{v} \cdot e_j(r_k),
\]
\[
S_j = \mathbf{M} : e_j^+(r_k),
\]
(\text{A4})

so that the normal-mode summation (A1) can be recast into a more manageable form as in (4), where \(\bar{R}_j = w_j R_j\). In Woodhouse & Girmans (1982), the elements \(R_j\) and \(S_j\) were expressed in terms of the generalized spherical harmonics. Here, for the sake of simplifying the final expressions of the 2-D kernels, we take a somewhat different approach and write \(R_j\) and \(S_j\) as ordinary spherical harmonics being operated upon by two surface differential operators \(I_k\) and \(E_k\):
\[
I_k = I_{k0} Y_m(\theta_k, \phi_k),
\]
\[
E_k = E_{k0} Y_m(\theta_k, \phi_k),
\]
(\text{A5})

with
\[
I_k = I_k + I_k^2 \frac{\partial}{\partial \theta_k} + I_k^2 \frac{\partial}{\partial \phi_k},
\]
\[
E_k = E_k + E_k^2 \frac{\partial}{\partial \theta_k} + E_k^2 \frac{\partial}{\partial \phi_k} + E_k^2 \frac{\partial^2}{\partial \theta_k^2} + E_k^2 \frac{\partial^2}{\partial \phi_k^2} + E_k^2 \frac{\partial^2}{\partial \theta_k \partial \phi_k},
\]
(\text{A6})

where
\[
I_k = v_j U_j(r_k),
\]
\[
I_k^2 = v_k V_k(r_k) - v_k W_k(r_k),
\]
\[
E_k^2 = [v_j U_j(r_k) + v_k V_k(r_k)] / \sin \theta_k,
\]
\[
E_k = M_{0k} U_k(r_k) + (M_{0k} + M_{4k} r_k^4) U_k(r_k),
\]
\[
E_k = \frac{1}{r_s} \left\{ M_{0k} U_k(r_k) - M_{4k} Z_k(r_k) \right\} + \frac{\cos \theta_k}{\sin \theta_k} \left\{ [M_{4k} V_k(r_k) + M_{0k} W_k(r_k)] \right\},
\]
(\text{A7})

\[
E_k = \frac{1}{r_s} \left\{ M_{0k} U_k(r_k) - M_{4k} Z_k(r_k) \right\} - \frac{\cos \theta_k}{\sin \theta_k} \left\{ [M_{4k} V_k(r_k) - M_{0k} W_k(r_k)] \right\},
\]
\[
E_k = \frac{1}{r_s} \left\{ M_{4k} V_k(r_k) - M_{0k} W_k(r_k) \right\},
\]
(\text{A8})

and
\[
Y_k(r) = r V_k(r) - V_k(r) + U_k(r),
\]
\[
Z_k(r) = r W_k(r) - W_k(r).
\]
(A9)

Here a dot over a radial eigenfunction represents the corresponding first derivative. So far, all the expressions in this appendix are derived for the geographical coordinate system, with \(r, \theta\) and \(\phi\) being the radius, co-latitude and longitude of a point in the Earth. In the derivation of the 2-D Fréchet kernels, without loss of generality, we can rotate the coordinate system to the path-specific coordinate system in which both the source and the receiver are in the equatorial plane. The expressions in (A6) can then be simplified as
\[
I_k = \delta \partial \frac{U_k(r_k) + [v_k V_k(r_k) + v_k W_k(r_k)]}{\cos \theta_k},
\]
\[
E_k = \delta \partial \frac{U_k(r_k) + [M_{0k} + M_{4k} r_k^4] U_k(r_k)}{\cos \theta_k}
\]
(\text{A9})

\[
\delta \partial \frac{M_{0k} V_k(r_k) + M_{0k} W_k(r_k)}{\cos \theta_k},
\]
\[
\delta \partial \frac{M_{4k} V_k(r_k) - M_{0k} W_k(r_k)}{\cos \theta_k},
\]
\[
\delta \partial \frac{2 M_{0k} V_k(r_k) + (M_{0k} - M_{4k}) W_k(r_k)}{\cos \theta_k},
\]

\[\text{APPENDIX B: RADIAL FUNCTIONS IN COUPLING MATRIX}\]

The normal-mode coupling matrix \(\delta Z\) due to general perturbations of the earth model was discussed in great detail in Woodhouse & Dahlen (1978) for an isolated multiplet based upon first-order perturbation theory. Woodhouse (1980) extended the coupling effect to modes belonging to different multiplets and provided formulae for the coupling matrix elements for isotropic earth models. Since then, many contributions to the derivation of \(\delta Z\) in various situations have been added by other researchers. Mochizuki (1986) and Tanimoto (1986) obtained the matrix elements for the perturbation of the general (anisotropic) elastic tensor. Henson (1989) derived
the expressions for $\delta Z$ for the discontinuity perturbation and for the ellipticity corrections in transversely isotropic earth models. All these results have been essential to the normal-mode theory based tomographic inversions.

One of the common features in the derivations of the normal-mode coupling matrix elements $\delta Z_{ij}$ has been the expansion of the earth’s lateral structure in terms of spherical harmonics. This reduces the calculation of the volume integral in $\delta Z_{ij}$ to the evaluation of a number of Wigner 3-j symbols in addition to the radial integral. While the spherical harmonics expansion of the lateral heterogeneity may be a convenient approach to deal with global tomographic inversions, it is not the case in regional studies where the interest is localized and higher resolution is desired. For this reason, in deriving the 3-D and 2-D Fréchet kernels in this study we do not express the model perturbations in any explicit fashion, and therefore the volume integral in the 3-D case and the plane integral in the 2-D case are retained. In this appendix, we provide expressions for the coupling matrix elements for the perturbations of the transversely isotropic elastic tensor, density and the topographies of discontinuities relative to a spherically symmetric and transversely isotropic reference model.

**Perturbation in the transversely isotropic elastic tensor**

For the perturbation of the general elastic tensor, the coupling matrix elements are defined as (e.g. Woodhouse & Dahlen 1978; Mochizuki 1986)

$$\delta Z_{ij} = \int e^*_j \cdot \delta C \otimes e_i \, d^3r, \quad (B1)$$

where $\delta C$ is the perturbation of the elastic tensor $C$ which has, in transversely isotropic models, five independent elements $L$, $N$, $C$, $A$, and $F$. Substituting the definition for the strain tensors $e_j$ (A2) into (B1) and after a certain amount of algebra, the integrand in (B1) becomes

$$e^*_j \cdot \delta C \otimes e_i = [\delta C \{U^r \otimes e^i \} + \delta A - \delta N r^{-2} X_i X_r + \delta F \{U^r \otimes 1 \}] Y^0_{lm} Y^0_{lm} + \delta L \{A^r \{U^r \otimes e^i \} + X_i X_r \} Y^{1+1}_{lm} Y^{1+1}_{lm} + \delta N \{A^r \{U^r \otimes e^i \} + X_i X_r \} Y^{3+2}_{lm} Y^{3+2}_{lm} + \delta N \{A^r \{U^r \otimes e^i \} + X_i X_r \} Y^{3+2}_{lm} Y^{3+2}_{lm} + \delta N \{A^r \{U^r \otimes e^i \} + X_i X_r \} Y^{3+2}_{lm} Y^{3+2}_{lm}, \quad (B2)$$

where $Y^0_{lm}$ are the generalized spherical harmonics (Phinney & Burridge 1973), functions $Y_i$ and $Z_k$ are defined in (A8), and

$$A^r = \sqrt{\frac{(l+1)}{2}}, \quad A^i = \sqrt{\frac{(l+1)(l-1)(l+2)}}{2}, \quad (B3)$$

$$X_k(r) = 2U_k(r) - l(l+1)\dot{V}_k(r).$$

Defining a vector $A^0_{lm} (r)$ whose elements are the radially dependent functions in the terms in (B2) for the individual elastic parameters:

$$A^0_{lm} (r) = (\{ A^0_{lm} (r) \}_1 \{ A^0_{lm} (r) \}_x \{ A^0_{lm} (r) \}_c \{ A^0_{lm} (r) \}_d \{ A^0_{lm} (r) \}_r \cdots)^T, \quad (B4)$$

the coupling matrix elements (B1) can be written in the form

$$\delta Z_{ij} = \int_0^\infty \left[ \sum_{M=-\infty}^{l} A^0_{lm} (r) Y^0_{lm} (0, \phi) Y^0_{lm} (0, \phi) \right] T \left[ \begin{array}{c} \delta m(r) \\ m_0(r) \end{array} \right] d^3r, \quad (B5)$$

where $[\delta m(r)/m_0(r)]$ is the symbolic vector introduced in (14) representing the model perturbations which now has the elements

$$\left[ \begin{array}{c} \delta m(r) \\ m_0(r) \end{array} \right] = \left( \frac{\delta L}{T} \frac{\delta N}{N} \frac{\delta C}{C} \frac{\delta A}{A} \frac{\delta F}{F} \cdots \right)^T. \quad (B6)$$

By inspection, we can easily write down the expressions for the elements of $A^0_{lm} (r)$:

$$\{ A^0_{lm} (r) \}_1 = -N r^{-2} X_i X_r \{ A^0_{lm} (r) \}_x = CU_i (r) U_j (r), \{ A^0_{lm} (r) \}_c = A r^{-2} X_i X_r \{ A^0_{lm} (r) \}_d = Ar^{-2} X_i X_r \{ A^0_{lm} (r) \}_r = \left. \frac{dU}{dr} \right|_{X_i X_r} \{ A^0_{lm} (r) \}_r = \left. \frac{dU}{dr} \right|_{X_i X_r}$$

$$\{ A^0_{lm} (r) \}_1 = -N r^{-2} X_i X_r \{ A^0_{lm} (r) \}_x = CU_i (r) U_j (r), \{ A^0_{lm} (r) \}_c = A r^{-2} X_i X_r \{ A^0_{lm} (r) \}_d = Ar^{-2} X_i X_r \{ A^0_{lm} (r) \}_r = \left. \frac{dU}{dr} \right|_{X_i X_r} \{ A^0_{lm} (r) \}_r = \left. \frac{dU}{dr} \right|_{X_i X_r}$$

which have the following property:

$$A^0_{lm} (r) = [A^0_{lm} (r)]^* \quad (B8)$$

Notice that we have defined both $A^0_{lm} (r)$ and $[\delta m(r)/m_0(r)]$ as open-ended vectors. New elements can be appended when perturbations in other types of model parameters, such as density and discontinuity topographies discussed next, are considered.

**Perturbation in density**

The coupling matrix elements $\delta Z_{ij}$ due to a perturbation in density can be expressed as (e.g. Woodhouse & Dahlen 1978)

$$\delta Z_{ij} = \int_0^\infty \rho [\{ e_i \cdot (V \Psi^T) \} + e^*_j \cdot (V \Psi^T) + 8\pi G \rho (\{ \dot{e}_j \} \cdot \{ \dot{e}_j \}) \{ \dot{e}_j \} \cdot \{ \dot{e}_j \}) + g \Gamma - \omega^2 \{ e_i \} \cdot \{ e_i \}] (\frac{\delta \rho}{\rho}) d^3r, \quad (B9)$$

where $G$ is Newton’s gravitational constant and $\omega$ is a reference frequency which is defined as the centroid of the eigenfrequencies of all the coupled normal modes (Woodhouse 1980). The functions $g(r)$ and $\Psi$ are the acceleration of gravity in the reference model and the change in gravitational potential associated with the $j$th mode, respectively, and

$$\Gamma = \frac{1}{2} \left\{ (\dot{e}_i \cdot [\dot{V} \{ \dot{e}_j \}]) + e^*_j \cdot [\dot{V} \{ \dot{e}_j \}] - (\{ \dot{e}_j \} \{ \dot{V} \cdot e_j \}) - (\{ \dot{e}_j \} \{ \dot{V} \cdot e_j \}) - \frac{4}{r} (\{ \dot{e}_j \} \{ \dot{V} \cdot e_j \}) \right\}. \quad (B10)$$
The change in gravitational potential $\Psi(r)$ can also be expanded in terms of spherical harmonics:

$$\Psi(p) = \psi_0(p) Y_m^p(\theta, \phi),$$

(B11)

where the radial function $\psi_0(r)$ can be related to the corresponding radial eigenfunctions $U_k(r)$ and $V_k(r)$ (Woodhouse & Dahlen 1978). With this arrangement, as in the case for the perturbation of the elastic tensor, the elements $\delta Z_{ij}$ can be written in the form

$$\delta Z_{ij} = \int_{\Theta} \sum_{M=-1}^1 \{A_{M}^{ij}(r)_{\rho} Y_{M}^{p}(\theta, \phi) Y_{M}^{p}(\theta, \phi) \left( \frac{\delta \rho}{\rho} \right) \} d^3 r,$$

(B12)

with

$$\{A_{M}^{ij}(r)_{\rho} = \frac{r}{2\rho} \left[ (8\pi G pr - 4G - \omega^2 r) U_k(r) + U_k(r) \Phi_k^p(r) + 2 \Phi_k^p(r) U_k(r), \right.$$ (B13)

$$+ [V_k(r) \mp i W_k(r)] \Phi_k^p(r) - 2\omega r[V_k(r) \mp i W_k(r)] [V_k(r) \pm i W_k(r)], \]$$

$$\Phi_k^p(r) = 2\psi_0(r) + g(r) U_k(r),$$

$$\Phi_k^p(r) = r^2 \psi_0(r) + \frac{1}{2} \Omega [r + 2g(r)V_k(r)].$$

(B14)

The elements $\{A_{M}^{ij}(r)_{\rho}$ and $\delta \rho/\rho \}$ can now be appended to the vectors in (B4) and (B6), respectively. Notice that the property in (B8) still holds for the density perturbation.

### Perturbations in the topographies of discontinuities

For the perturbation in the topography of a discontinuity with radius $r_d$ in the reference model, the normal-mode coupling matrix elements $\delta Z_{ij}$ are expressed as (e.g. Woodhouse & Dahlen 1978; Henson 1989)

$$\delta Z_{ij} = \int r_d \sum_{M=-1}^1 \left\{ [\tilde{\mathbf{F}} \cdot (\mathbf{C} \cdot \mathbf{e}_j)] + [\tilde{\mathbf{F}} \cdot (\mathbf{C} \cdot \mathbf{e}_j) \cdot \mathbf{e}_d] \right\}$$

$$- \rho \left[ \mathbf{e}_j \cdot (\nabla \mathbf{e}_j) \right] + 8\pi G \rho \mathbf{e}_j (\mathbf{r} \cdot \mathbf{e}_d)$$

$$+ \mathbf{G} \mathbf{e}_j + \mathbf{e}_j \cdot \mathbf{e}_d$$

$$+ (\mathbf{e}_d \cdot \mathbf{e}_j) \cdot (\mathbf{r} \cdot \mathbf{e}_d) - (\mathbf{r} \cdot \mathbf{e}_d),$$

(B15)

where the notation $\| f(r_d) \|$ is defined as $f(r_d) + f(r_d - )$, the difference in the values of $f$ on the two sides of the discontinuity $r = r_d$. On the top side of the Earth's free surface, $r = a +$, all the quantities except for the radius $r_d$ and the acceleration of gravity $g(a +)$ are assumed to vanish. The integral is over $\Sigma_2$, the spherical surface of the discontinuity. These elements can still be written in the form

$$\delta Z_{ij} = \int_{\Sigma_2} \sum_{M=-1}^1 \{A_{M}^{ij}(r)_{\rho} Y_{M}^{p}(\theta, \phi) Y_{M}^{p}(\theta, \phi) \left( \frac{\delta \rho}{\rho} \right) \} d^2 r_d,$$

(B16)

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the direction normal to this plane with a stationary-phase approximation. The expression for the 2-D kernel can be further simplified by rotating the geographical coordinate system to the path-specific coordinate system in which both the source and the receiver are in the equatorial plane with longitudes of 0 and \( \Delta \), respectively. Using the expressions for the operators \( \tilde{I}_k \) and \( \tilde{E}_k \) in (A9) for this coordinate system, the 2-D kernel can be written in the form of (26). The explicit expressions for the coefficients \( C_{kk} \) and \( S_{kk} \) are

\[
C_{kk} = \varepsilon U_k(r_k) \left[ M_{m} U_k(r_k) + (M_{mh} + M_{ww}) r_s^{-1} U_k(r_k) \right]
- M_{ww} r_s^{-1} \left( l + \frac{1}{2} \right)^2 V_{l}(r_s)
+ \varepsilon \left( l + \frac{1}{2} \right) V_{l}(r_k) M_{ww} r_s^{-1} \left( l + \frac{1}{2} \right) V_{l}(r_s),
\]

\( \text{(C1)} \)

\[
S_{kk} = \varepsilon U_k(r_k) M_{ww} r_s^{-1} \left( l + \frac{1}{2} \right) Y_{l}(r_s) - \varepsilon \left( l + \frac{1}{2} \right) V_{l}(r_k)
\times \left[ M_{m} U_k(r_k) + (M_{mh} + M_{ww}) r_s^{-1} U_k(r_k) \right]
- M_{ww} r_s^{-1} \left( l + \frac{1}{2} \right)^2 V_{l}(r_s),
\]

\( \text{(C2)} \)

for the spheroidal modes or the \( P-SV \) waves, and

\[
C_{kk} = \varepsilon \left( l + \frac{1}{2} \right) W_{l}(r_k) M_{ww} r_s^{-1} \left( l + \frac{1}{2} \right) Z_{l}(r_s),
\]

\( \text{(C3)} \)

\[
S_{kk} = \varepsilon \left( l + \frac{1}{2} \right) W_{l}(r_k) M_{ww} r_s^{-1} \left( l + \frac{1}{2} \right)^2 W_{l}(r_s),
\]

\( \text{(C4)} \)

for the toroidal modes or the \( SH \) waves.